The Structure of the American Economy

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Abstract

We explore the relationship between input-output accounts and the national revenue function. The generalized inverse of an economy's technology matrix carries information relating changes in endowments with changes in outputs; its transpose relates output prices and factor prices. Our primary theoretical contribution is to derive an economy's revenue function for an arbitrary Leontief technology. Our main empirical contribution is to compute the national revenue function for the American economy in 2003 and to describe its properties. We implement our ideas using two different models: one where all factors are mobile and another with capital specific to each sector in the economy.

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1. Introduction

Presenting a completely novel approach to analyzing the supply side of an economy, we show that an input-output table contains all the information needed to describe an economy's Rybczynski matrix. These effects relate marginal increases in an economy's resources with marginal changes in its vector of net outputs, when goods prices and thus factor uses are fixed. The same information can be interpreted as Stolper-Samuelson effects: the link between output prices and factor rewards, when endowments are in fixed net supply. We develop the theory, present leading examples, and implement our ideas using data from the American economy in 2003.

There is no immediate antecedent to our work. Still, we stand on the shoulders of three giants. First, we would not have begun this work if Leontief (1951) had not devised input-output accounting; indeed the title of our paper pays blunt homage to his influence. Second, we bring the powerful mathematical tools that Moore (1920) and Penrose (1955) developed to applied general equilibrium theory. These authors created a technique to characterize all the solutions to a system of linear equations, even when the set of equations is underdetermined or only "approximately" correct. Their generalized inverse is readily available in most modern statistical software, and we use it to implement our ideas. Third, we characterize the supply side of an economy using the national revenue function. For a fixed technology, the national revenue function maps an economy's endowments and an output price vector into the maximum revenue attainable by the economy. Its Hessian—the matrix of cross-partial derivatives of revenue with respect to output prices and net factor supplies—is an economy's Rybczynski matrix. The national revenue function and the dual approach to general equilibrium theory had a profound influence on a generation of international economists. Dixit and Norman (1980) give its most elegant exposition.

Our work grows out of the mainstream of international trade theory, but it falls squarely within the bailiwick of modern macroeconomics. Although we construct a theoretical foundation that has general applications, we are really interested in the details of the American economy. The theoretical framework we develop is designed to answer this kind of question: What effect does an increase in the price of refined petroleum have on the wage of unskilled labor? We are also able to analyze how immigration of unskilled labor or the accumulation of capital will influence the vector of net outputs of the American economy when one controls for the output prices. We hope our techniques will have wide applications in international economics and in macroeconomics.

Our main theoretical contribution is to derive the revenue function for an arbitrary Leontief production structure. This function is smooth with respect to its two arguments, endowments and output prices. Hence, it is very well behaved, its properties are easy to describe and to compute, and one can compute the exact Rybczynski derivatives for any economy that reports an input-output table and conformable data on factor uses. Our approach fills in a gap in the literature: when there are more goods than factors, the Rybczynski effect is indeterminate because the revenue function is not differentiable with respect to prices. Our approach allows one to compute completely the sub-gradient characterizing the supply correspondence.

We make two broad empirical contributions. First, we look at the United States economy in 2003 disaggregated into 63 sectors and six factors, capital and five broad types of labor, that are mobile across all sectors. We show for example that capital has its strongest positive Rybczynski affect on real estate and its strongest negative such effect on “Computer systems design and related services.” We are also able to estimate the shadow value for each of the six factors, and we demonstrate that capital’s gross rate of return in the American economy was 13.6% in 2003. Our second application is based upon a Ricardo-Viner model in which capital is specific to each sector and the mobile factors are five broad types of labor. Here are three desultory examples that might whet the reader’s appetite for the empirical analysis to come. The sector whose price has the strongest positive effect on the reward to professional occupations is “Computer systems design and related services,” and the sector whose price has the strongest negative effect on the reward to professional occupations is “Legal services.” A million dollar increase in the price of refined petroleum has its strongest negative effect on the specific factor used in “Truck transportation.” Such an oil price shock lowers the reward to a stock of $1 million of capital specific to the Truck and Transportation sector by $128 thousand.

Our real empirical contribution is to show how easy it is to implement our theory and to derive plausible empirical effects that describe the details of the American economy. Since our theory is based upon an arbitrary Leontief structure, it can handle any degree of aggregation and any model that the researcher might find appealing. Our theory is simple, so its applications are broad.

The rest of this paper is structured as follows. The second section is a brief review of the literature; we recognize that we are trying to revive a field of research that has fallen into desuetude, so we will emphasize the novelty of our work is and how it brings together branches of international economics and macroeconomics. The third section develops the theory that is the foundation for our empirical work. It has five subsections:

(1) a summary of the revenue function; (2) a review of the factor pricing equations; (3) a statement of the properties of the Moore-Penrose generalized inverse; (4) a statement of our main theorem showing that the transpose of this generalized inverse is indeed the Rybczynski matrix for an economy with a Leontief technology; and (5) a discussion of how our analysis extends to more general technologies. The fourth section explores in depth three leading examples: (1) the Solow model, the simplest case with more factors than goods; (2) the Ricardian model, the simplest case where there are more goods than factors; and (3) the classic two-sector model in trade theory where there are an equal number of goods and factors. The fifth section applies our ideas to the American economy in 2003. We implement empirical analyses of two different models: (1) one where all six factors are mobile; and (2) a second where each sector has sector-specific capital and there are five types of mobile labor. This section shows that our theoretical analysis is quite plausible, and it serves as an illustration of how powerful and simple our theory is to implement in realistic empirical applications. It also examines the model's fit for the case where there are six mobile factors. The sixth section summarizes our contributions and gives suggestions for future research.

2. Brief Review of the Literature

Input-output accounting was invented to calculate the necessary resources for a marginal increase in an economy's output. Trade theory explores how a marginal increase in resources affects an economy's outputs. Hence input-output accounting and international trade look at similar phenomena from opposite ends, and it is not surprising that the same data shed light on both lines of inquiry. The (rectangular) matrix of direct and indirect factor requirements has been the central tool of input-output analysis for half a century, but no one has ever noticed that its generalized inverse is the Hessian of the revenue function for an economy with a Leontief technology.

There is a large literature in empirical trade that explores the relationship between endowments and outputs. Using flexible forms, Kohli (1991) and Harrigan (1997) estimate national revenue functions; both tie theory and empiricism together carefully. These estimates are an important step in describing the sources of comparative advantage. Fitzgerald and Hallak (2004) and Schaur, Xiang, and Savikhin (forthcoming) estimate reduced form Rybczynski equations. Fitzgerald and Hallak claim that failure to control for productivity differences produces biased estimates, and Schaur, Xiang, and Savikhin state that the average effects across all local industries are positive. Our work renders all this statistical analysis moot; we show how to *compute—not estimate*--exact Rybczynski derivatives for any country that has an input-output table and conformable uses of factors. We find much richer patterns than those in the literature and reap an overlooked bonus: our exact effects are also Stolper-Samuelson (directional) derivatives.

The notion that shocks to the sectors of an economy may be a central cause of business cycles may be as old as Quesnay (1758). Black (1982) breathed new life into this idea, and Long and Plosser (1983) give an elegant exposition of a model where sectoral shocks cause aggregate economic fluctuations. The example that forms the crux of Long and Plosser's article shows that positive co-movements arise because of the income effects inherent in a shock to any one sector. It is striking that business cycles occur even when labor is the only factor of production. Still, a model of real business cycles without capital is quite limited, and when there is more than one factor, there is no simple relationship between shocks to one sector and positive co-movements in all sectors. The Rybczynski effects for capital that we identify have a ready interpretation as the supply side effects of wealth accumulation when output prices are held fixed.

The best reason for resuscitating an older literature is to show that inconsistencies in measuring the factor content of trade may well plague much of the research on the factor content of trade in the last two decades. It is not well known that input-output accounts can be also be used to derive relationships between factor prices and goods prices. We exploit this relationship to a much greater extent than has ever been done before. A large body of work in international economics uses data on intermediate inputs from one source and direct factor requirements from other sources to compute the factor content of trade. We have shown (2008a and 2008b) that measurement error afflicts much of this research. Our theory and the statistical tests substantiate that improper measurement—most likely in the direct uses of capital—is a serious concern for anyone using input-output analysis to compute factor content. The whole literature ought to use consistent data if it wants to give Heckscher-Ohlin-Vanek theory a fair chance.

3. Theory

A. The Revenue Function

Let v be the $f \times 1$ vector of aggregate inputs of primary factors that are in fixed supply and y be the $\ell \times 1$ vector of outputs. Technological considerations are summarized by a set of feasible combinations of outputs and inputs

$$F \subset \mathbb{R}^{\ell+f} .$$

It is convenient to assume that this set is compact for fixed inputs v. Producers take the $\ell \times 1$ vector of output prices p as given. The revenue function is

$$r(p,v) = \max_{y} \{p^T y | (y,v) \in F\}$$

The main theoretical advantage of the national revenue function is that it allows one to summarize the relationship between endogenous and exogenous variables succinctly.

For example, the output vector is the gradient of the national revenue function: [1]

$$y^T = r_p(p, v).$$

The revenue function is appealing because it is so general. It is homogeneous of degree one in prices; thus $r_p(p, v)$ is homogenous of degree zero in prices. It follows that $r(p, v) = r_p(p, v)p$ and $r_{pp}(p, v)p = 0$. Also $r_v(p, v)$ gives the derived inverse demand for factors of production. If factors are in fixed supply, their prices are:

$$w^T = r_v(p, v).$$

The function $r_v(p, v)$ is also homogeneous of degree one in p.

The Rybczynski matrix

$$r_{pv}(p, v) = \begin{bmatrix} \partial^2 r/\partial p_1 \partial v_1 & \cdots & \partial^2 r/\partial p_1 \partial v_f \\ \vdots & \ddots & \vdots \\ \partial^2 r/\partial p_\ell \partial v_1 & \cdots & \partial^2 r/\partial p_{1\ell} \partial v_f \end{bmatrix}$$

is the focus of this paper. Its canonical element $\partial^2 r/\partial p_i \partial v_j$ shows how the output of good i changes with respect to a marginal increment in the endowment of factor j, if one holds factor prices and thus factor requirements constant.

The transpose of the Rybczynski matrix shows the Stolper-Samuelson effects:

$$r_{vp}(p, v) = r_{pv}(p, v)^T$$

[1] In this paper, all gradients are row vectors. We use the notation $r_p(p, v) = [\partial r/\partial p_1 \quad \cdots \quad \partial r/\partial p_\ell]$. In this sub-section for ease of exposition, we are quite blithe in assuming that all functions are differentiable. They typically are not, and that is why this elegant theoretical approach has had limited practical appeal for empiricists. We will show below how to compute the sub-gradient that is the supply correspondence when there are more goods than factors and also how to compute the complete set of factor prices consistent with perfect competition when there are more factors than goods.

Each element of this matrix describes the marginal effect of a change in the price of good j on the reward to factor i. Since factor rewards are homogenous of degree one in prices, the Stolper-Samuelson effects satisfy an important restriction:

$$w = r_{vp}(p, v)p$$

This equation states that the sums of the Stolper-Samuleson effects, weighted by the price of output in each sector, are the shadow values of the factors in the economy, a fact that we will use in our empirical analysis. If there are constant returns to scale, then outputs are homogenous of degree one in v. In this case, the Rybczynski matrix also satisfies:

$$y = r_{pv}(p, v)v$$

This equation states that the sums of the Rybczynski effects, weighted by the quantities of the economy's fixed endowments, are the elements of economy's output vector. Again, we will use this fact that in our empirical work below.

B. The Factor Pricing Equations and the Resource Constraints

The usual relationship between factor prices and goods prices is given by:

$$Aw \geq p, y \geq 0, with\ complementary\ slackness$$

where a_{ij} is the unit input requirement of factor j in the output of good i. Because we will be interested in marginal changes in our empirical work, we will restrict our attention to strict equalities without loss of generality. If the $i-th$ good's unit cost exceeds its price, then it will not be produced. Then we will set $a_{ij} = 0$ for $j = 1, \ldots, f$ and also write $p_i = 0$. In this case, the following equality is true:

$$Aw = p$$

Any factor rewards w that solve this modified system will also satisfy the original equations, and any solution of the original system will give factor prices that also solve the modified system. Also, since $y_i = 0$ in the original system, the modified technology matrix will automatically satisfy the resource constraints in the original system.

The full employment equations are:

$$A^T y \leq v, w \geq 0, with\ complementary\ slackness.$$

If the $j-th$ factor is in excess supply, then its reward $w_j = 0$. Now we set $a_{ij} = 0$ for $i = 1, \dots, \ell$ and also write $v_j = 0$. Then

$$A^T y = v,$$

and each solution to this modified system corresponds to a solution in the original one. Likewise, every vector of outputs in the original system will solve the modified one.

The national revenue function can also be defined as the minimum value of payments to factors of production that is consistent with the zero-profit conditions:

$$r(p, v) = \min_{w}\{w^T v | Aw \geq p\}$$

This approach is helpful if one is interested in using Shephard's Lemma to derive aggregate factor demands. For example, it predicts that factor prices are given by the gradient of the unit isoquant evaluated at the endowment vector in a model with one good and several factors. In our empirical work, we use the fact that fixed factor prices entail a restriction on admissible endowment changes if there are more factors than goods.

C. The Moore-Penrose Pseudoinverse

Let A be an $\ell \times f$ matrix. Then its Moore-Penrose pseudoinverse is the unique $f \times \ell$ matrix A^+ that satisfies these four properties:

(P1) $AA^+A = A$;

(P2) $A^+AA^+ = A^+$;

(P3) $A^+A = (A^+A)^T$; and

(P4) $AA^+ = (AA^+)^T$;

If A is square and has full rank, then $A^+ = A^{-1}$. If $A^T A$ has full rank, then $A^+ = (A^T A)^{-1} A^T$ can be computed easily. Every matrix has such a pseudoinverse.[2]

The primary advantage of the Moore-Penrose pseudoinverse is that it gives the complete set of solutions to the system of equations $Ax = b$. This set is:

$$x = A^+ B + (I - A^+ A) z$$

where z is an arbitrary $f \times 1$ vector. In fact, this pseudoinverse even gives a solution to an over-determined and inconsistent system $Ax \approx b$. Then $x = A^+ b$ is the vector of coefficients of the least squares estimates from the regression of b on the columns of A.

If a row of a non-null matrix A consists of zeros, then the corresponding column of A^+ does also. This fact justifies our restrictions that $Aw = p$ and $A^T v = y$ hold with equality, as long as one works with a modified matrix that replaces the appropriate row or column of the original technology matrix with zeros whenever a constraint is slack.

D. A^+ is the Stolper-Samuelson Matrix and ${A^+}^T$ is the Rybczynski Matrix

The production function for a fixed coefficients technology is:

$$y_i = \min \{v_{i1}/a_{i1}, \ldots, v_{if}/a_{if}\},$$

where v_{ij} is the input of factor j into sector i. Let A be the $\ell \times f$ matrix of (direct and indirect) factor requirements that are observed in the data. Assume that the $\ell \times 1$ vector of output prices p is given. Then the complete solution for factor prices is:

$$w = A^+ p + (I - A^+ A) z$$

where z is an arbitrary $f \times 1$ vector. The matrix $I - A^+ A$ projects z onto the null space of A. This expression gives all factor prices consistent with perfect competition.

[2] Albert (1972) gives a very good exposition of the properties of the Moore-Penrose generalized inverse.

Since factor payments exhaust revenues,

$$r(p,v) = v^T w = v^T A^+ p + v^T (I - A^+ A) z.$$

The full employment conditions imply that the endowment vector is in the row space of the technology matrix. Since this space is the orthogonal complement of its null space, $v^T (I - A^+ A) z = 0$ for any z.[3] Then factor prices are:

$$w = r_v(p,v) = A^+ p + (I - A^+ A) z.$$

where z is arbitrary. This expression gives the *set of all factor prices* that are consistent with the zero-profit conditions. It is typical in the literature to explain that factor prices are not tied down when there are more factors than goods and that they are derived from other extraneous considerations--such as the full employment conditions--that have nothing to do with unit costs. Of course, that argument does not work for the case of a fixed coefficients technology or for any other where each output is not differentiable with respect to every input.

It is constructive to derive the national revenue function in an analogous manner from the economy's resource constraint:

$$A^T y = v.$$

The complete solution for the output correspondence is:

$$y = (A^T)^+ v + (I - (A^T)^+ A^T) z$$

where z is now an arbitrary $\ell \times 1$ vector. Since the value of output is national revenue,

$$r(p,v) = p^T y = p^T (A^T)^+ v + p^T (I - (A^T)^+ A^T) z$$

[3] A simple way to see this fact is to note that $v^T A = y^T$ and $A(I - A^+ A) = 0$.

Since $I-(A^T)^+A^T$ projects onto the null space of A^T and prices p lie in the column space of A, we conclude that $p^T(I-(A^T)^+A^T)=0$. Since $(A^T)^+=(A^+)^T$, this formula is simply the transpose of the one derived using the income approach.

These results are significant enough to state formally.

Theorem: *Consider an economy with a Leontief technology. Assume that all resources are fully employment and that all good are produced. Let p be the $\ell \times 1$ vector of goods prices, v be the $f \times 1$ vector of factor endowments, and A be* the *$\ell \times f$ matrix of unit input requirements. Then the revenue function is the quadratic form (1).*

$$r(p,v)=v^TA^+p+v^T(I-A^+A)z_f+z_\ell{}^T(I-AA^+)p \qquad (1)$$

where $z_f \in \mathbb{R}^f$ and $z_\ell \in \mathbb{R}^\ell$ are arbitrary.

Proof: *The full employment condition implies that $v^T(I-A^+A)=0$, and the zero-profit condition implies that $(I-AA^+)p=0$. Hence the particular solution in (1) is the value of national revenue since each quadratic form involving a homogenous term has value zero. By construction, the gradient of (1) with respect to p gives the gives the supply correspondence, and its gradient with respect to v gives the set of factor prices consistent with the zero-profit conditions. Further, (1) satisfies all the requisite homogeneity restrictions with respect to its two arguments. Q.E.D.*

The theorem has two immediate implications. If there are at least as many goods as factors and the technology matrix has full (column) rank, then $I-A^+A=0$ and factor prices are completely determinate. If there are at least as many factors as goods and the technology matrix has full (row) rank, then $I-AA^+=0$ and the output vector is determinate. There are interesting cases—of significant empirical relevance—where the technology matrix does not have full rank. For example, there are many models in macroeconomics with more (differentiated) goods than factors, but all sectors have identical factor intensities. Then the first homogeneous term in (1) is not null, and the gradients of the national revenue function show all factor prices that satisfy the zero-profit conditions, and it also gives the entire supply correspondence. Likewise, there are some models with at least as many factors as goods where the technology matrix does not have full rank; this situation arises when at least two sectors have identical factor intensities. In this case, the gradients of (1) again give the supply and factor-price correspondences.

The theorem shows that the national revenue function is (infinitely) differentiable with respect to both of its arguments, output prices and endowments. From our perspective, the most important of its implication is the following corollary.

Corollary: *Under the Theorem's assumptions, the economy's Stolper-Samuelson matrix is A^+ and its Rybczynski matrix is A^{+T}.*

This result is controversial at first blush. Assume there are strictly more goods than factors. Trade theorists allege that the Rybczynski effect is not defined in this case. But it is obvious that the only part of the supply correspondence that depends upon endowments is the particular solution in (1). Fix output prices and thus factor uses and consider a marginal change in endowments. We will show in Section 4 that any resultant change in outputs can be decomposed into two parts: (1) a change that is orthogonal to the economy's production possibility frontier; and (2) a movement along one of its flats. Only the former has any effect on national revenue, and that is why it is properly defined as the Rybczynski effect. Indeed, the beauty of (1) is that it solves the indeterminacy that has plagued the empirical literature.

The typical regression that estimates a "Rybczynski effect" actually captures the demand-side effects of wealth changes. Some of the more careful researchers note that the Rybczynski effect is indeterminate in the usual case where there are many goods and few factors. Some assume an (infinitely differentiable) translog approximation to the national revenue function and estimate its parameters, imposing symmetry and homogeneity restrictions. In essence, one notes the problem in theory and then assumes it away blithely in empirical work. Not only are such regressions based upon a potentially misleading approximation; they are not even identified in theory.

There is an even better reason to identify ${A^+}^T$ as *the* Rybczynski matrix. Its transpose gives the unique solution for factor prices $w = A^+p,$ where we have used that $I - A^+A = 0$ when $\ell \geq f$ and the technology matrix has full rank. Since all goods are produced, $Aw = p$. In this case, $w = A^+p = A^+Aw$ is an identity. It is obvious that A^+ is the only candidate for a Stolper-Samuelson matrix when input requirements are such that several sectors are active. (This is exactly what one would expect in the long run when local techniques adjust so that several sectors are competitive at prevailing world prices.) Hence, defining ${A^+}^T$ as the Rybczynski matrix maintains the duality between Rybczynski and Stolper-Samuelson effects at the heart of classical trade theory!

E. Marginal Changes in Output Prices and Factor Endowments

Consider a more general technology where the unit input requirements depend upon factor prices. This technology is described by an $\ell \times f$ matrix $A(w)$. Let dp a vector of marginal changes in goods prices and dw be the corresponding vector of changes in factor rewards. Then the following system of equations is true:

$$A(w)dw = dp. \tag{2}$$

Equation (2) uses the envelope theorem: cost minimization entails that marginal changes in factor uses evaluated at the original factor prices incur no incremental cost. Thus the logic of the Stolper-Samuelson theorem entails that every technology acts locally like one with fixed coefficients.[4] There is one important qualification that must be stated explicitly: marginal changes in prices must lie in the column space of the matrix $A(w)$. If there are more goods than factors, or if $A(w)$ is not of full rank, then there are explicit restrictions on how goods prices can change.

If there are more factors than goods and the technology matrix has full rank, then output prices are free to move in any direction. In this case, $A(w)^+$ is the Stolper-Samuelson matrix for fixed endowments that maintain full employment. If there are fewer factors than goods or the technology matrix does not have full rank, then the Stolper-Samuelson matrix gives the components of a directional derivative that map price changes--restricted to the column space of $A(w)$--onto changes in factor rewards.

Now fix output prices and thus factor prices and unit input requirements. Then the logic of the Rybczynski theorem entails:

[4] Jones (1965) reminded us that the underlying Leontief production functions stands in for every possible neo-classical production structure if one is only considering local changes in goods prices.

$$A(w)^T dy = dv. \quad (3)$$

Equation (3) imposes two conditions: first, factor prices are fixed; and second, changes in endowments must lie in the *row* space of $A(w)$.

If there are more factors than goods and the technology matrix has full (column and thus row) rank, then endowments changes are restricted to lie in the row space of the technology matrix. Then the elements of the Rybczynski matrix are components of a directional derivative that show how the restricted endowment changes map onto output changes when factor prices are fixed. If there are fewer factors than goods or the technology matrix does not have full rank, then $A(w)^{+T}$ is the economy's unfettered Rybczynski matrix.

One of these two restrictions is usually moot. For example, if there are more goods than factors and the technology matrix has full rank, then endowments are free to move in any direction, and we prefer to interpret $r_{pv}(p,v) = A^{+T}$ as a Rybczynski matrix for fixed goods prices that lie in the column space of A. If there are more factors than goods and the technology matrix has full rank, then output prices are free to move in any direction, and we prefer to interpret $r_{vp}(p,v) = A^{+}$ as a Stolper-Samuelson matrix for fixed endowments that maintain full employment.

4. Three Leading Examples

We sketch out the three simplest examples that illustrate the underlying theory.

Example 1: The Solow model is the simplest case where the number of factors exceeds the number of commodities. The vector of endowments is $v = [K \quad L]^T$, and technology is described by an aggregate production function $Y = F(K, L)$ that exhibits constant returns to scale. The unit input requirements depend upon factor prices:

$$A(w,r) = [a_K(w,r) \quad a_L(w,r)]^T,$$

where w is the wage rate and r is the rentals rate. Of course, factor prices are not even locally independent of endowments. The Stolper-Samuelson matrix is:

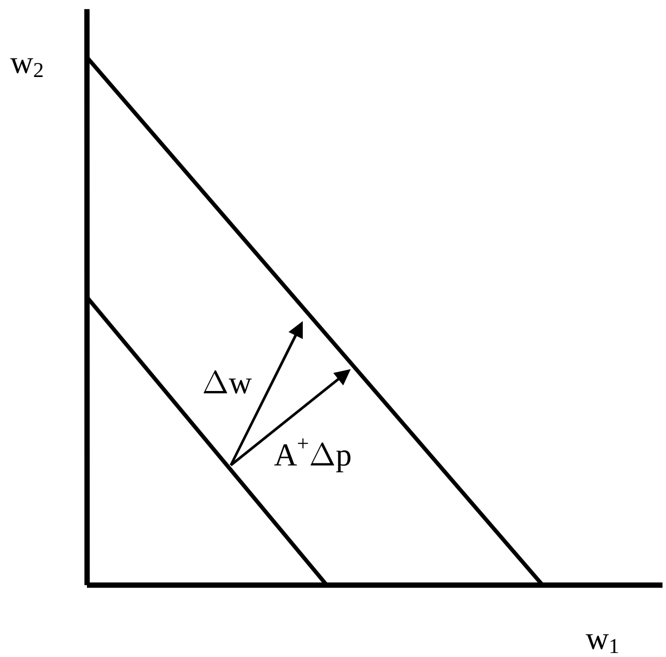
$$A^+ = \begin{bmatrix} a_K/(a_K^2 + a_L^2) \\ a_L/(a_K^2 + a_L^2) \end{bmatrix},$$

where we have suppressed the dependence on factor prices for notational convenience. Three points are in order. First, for fixed endowments, this matrix allows any change in output prices (in the trivial one-dimensional space in which they lie). Second, the Stolper Samuelson matrix *does not* consist of the marginal products of capital and labor; it is instead collinear with the average products of these factors. Third, within the strict framework of a Leontief technology where aggregate output $F(K,L) = \min\{K/a_K, L/a_L\}$, the Stolper-Samuelson matrix can be construed as a theory of factor prices. Among all strictly positive factor rewards that satisfy the zero-profit conditions $\{(w,r) \in \mathbb{R}_+^2 | p = a_K r + a_L w\}$, it picks the wage-rentals ratio a_L/a_K that corresponds with the economy's aggregate capital-labor ratio.

Figure 1 shows the Stolper-Samuelson effects in this case. The horizontal axis measures the first factor price and the vertical axis measures the second one. Let the price of aggregate output change by an arbitrary amount Δp. Then any observed change in factor prices $\Delta w = A^+\Delta p + u$ can be decomposed into two orthogonal parts. The first part $A^+\Delta p$ is orthogonal to the unit cost functions, and it the only direction that affects national income. The second part u (not drawn) has no effect on aggregate factor costs and thus no impact on national income. The first part comes from the particular solution in (1), and u lies in the linear space defined by the first homogenous term in that equation. In the empirical analysis in Section 4, we use the properties of the Stolper-

Samuelson derivatives to compute the shadow values of factors in the national economy $w = r_v(p,v)^T = A^+p$ in a model where there are more goods than factors. We are able to compute these shadow values even though output prices are not observable in our data.

Figure 1: Stolper-Samuelson Effects



The interpretation of the Rybczynski matrix A^{+T} in this case is subtle. Endowments are constrained to lie in the linear subspace generated by the economy's capital-labor ratio. Only marginal changes $[dK/K \quad dL/L]^T$ of equal proportions can maintain full employment at the factor prices that are assumed fixed. Then the elements of the Rybczynski matrix are components of a directional derivative that explain the change in aggregate output by attributing weights $K^2/(K^2+L^2)$ and $L^2/(K^2+L^2)$ to the changes in capital and labor respectively.

Example 2: The Ricardian model is the simplest case where the number of goods exceeds the number of factors. The vector of endowments is simply $v = L$. Technology is summarized by the production possibility frontier $\{(y_1, y_2) \in \mathbb{R}_+^2 | a_1 y_1 + a_2 y_2 = L\}$, where a_i is a sector's labor coefficient.

Let $A = [a_1 \quad a_2]^T$ be unit labor requirements observed in the data. The Stolper-Samuelson matrix is:

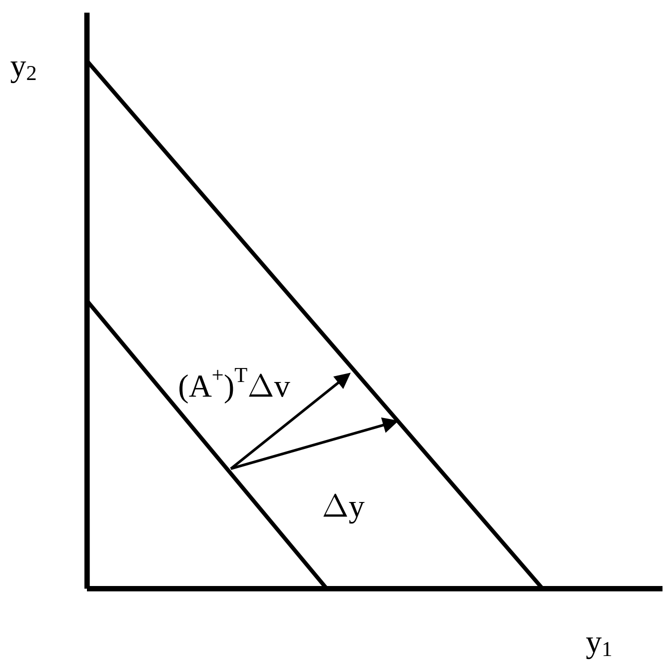
$$A^+ = [a_1/(a_1^2 + a_2^2) \quad a_2/(a_1^2 + a_2^2)\ ]^T.$$

Now its interpretation is subtle since output prices are constrained to lie in the linear subspace generated by A. Only marginal changes of equal proportions $[dp_1/p_1 \quad dp_2/p_2]^T$ can assure positive outputs of both goods. Then the elements of the Stolper-Samuelson matrix are components of a directional derivative that explain the change in the wage rate by giving weights $p_1^2/(p_1^2 + p_2^2)$ and $p_2^2/(p_1^2 + p_2^2)$ to the changes in the prices of the first and second good respectively.

Endowments are free to move in any (trivial) direction, but the Rybczynski A^{+T} matrix chooses one element of the supply correspondence. Indeed, only movements in the direction A^{+T} affect national revenue. In any other direction, a feasible change in outputs trades off one good against another according to the fixed marginal rate of transformation inherent in this economy. This tradeoff has no effect on revenue.

Figure 2 shows the Rybczynski effects in this case. Now the horizontal axis measures output of the first good and the vertical axis measures that of the second one. Let the endowment of labor change by an arbitrary amount Δv. Then any observed change in output $\Delta y = A^{+T}\Delta v + u$ can be decomposed into two orthogonal components. The first part $A^{+T}\Delta v$ is orthogonal to the economy's production possibility frontier, and it is the only direction that affects national revenue. The second part u (again not drawn) has no effect on the value of output and thus no impact on GDP; it lies in the linear space defined by the second homogenous term in (1). In Section 4, we use the properties of the Rybczynski derivatives to predict the output effects of an arbitrary endowment vector $y = r_p(p, v)^T = {A^+}^T v$ in an economy where there are more goods than factors. We show that the model predicts remarkably well, both in and out of sample.

Figure 2: Rybczynski Effects



Example 3: The Leontief model with two goods and two factors is the simplest example of the classic "even" case in trade theory. The vector of endowments is again $v = [K \quad L]^T$ and technology is described by the production possibility set:

$$\{(y_1, y_2) \in \mathbb{R}_+^2 | a_{1K}y_1 + a_{2K}y_2 \leq K \text{ } and \text{ } a_{1L}y_1 + a_{2L}y_2 \leq L\},$$

where we are following the usual notation. We assume that $\{a_{1K}/a_{1L} \leq K/L \leq a_{2K}/a_{2L}\}$; thus the economy can produce both goods under full employment. Now

$$A = \begin{bmatrix} a_{1K} & a_{1L} \\ a_{2K} & a_{2L} \end{bmatrix}.$$

Let $m = \min\{a_{1K}/a_{1L}, a_{2K}/a_{2L}\}$ and $M = \max\{a_{1K}/a_{1L}, a_{2K}/a_{2L}\}$. Assume that $m \leq p_1/p_2 \leq M$ and thus both factors have strictly positive rewards. If A has full rank, then changes in endowments are not restricted. Also, A^T will have full rank, and thus changes in goods prices are not restricted. In this case, $A^+ = A^{-1}$ and the properties of this Stolper-Samuelson matrix are well understood. For example, it has a negative element in each column.

If A does not have full rank, then either it is trivial ($A = 0$) or it has rank one. If it is not trivial, then $A^+ = A^T/\|A\|^2$, where $\|A\| = \sqrt{\sum_i \sum_j a_{ij}^2}$. In this case the column spaces of A and A^T both have rank one, and $m = M$ since the two rows of A are collinear. Hence the economy's production possibility frontier is linear and output prices are tied down by the marginal rate of technical substitution, just as in the Ricardian model. Since each sector uses factors in identical proportions, the full employment conditions determine the admissible direction for endowment changes, just as in the Solow model.

These facts imply that A^+ can be interpreted as a Stolper-Samuelson matrix only for price changes that maintain the fixed ratio $m = p_1/p_2 = M$. In fact, this price ratio is the one at which there is a factor intensity reversal for an economy with a general technology matrix $A(w, r)$. Likewise, the capital-labor ratios in each sector are identical, and only changes in the economy's endowments keep the capital-labor ratio in the one-dimensional subspace spanned by $a_{1K}/a_{1L} = K/L = a_{2K}/a_{2L}$ Again, for an economy with a more general technology matrix $A(w, r)$, this is the unique capital labor ratio that characterizes both sectors.

Hence the Stolper-Samuelson matrix is an array that maps the components of the directional derivative of price changes onto factor price changes. Now the weights of these four components depend upon the relative prices of the two goods *and* the economy's aggregate capital-labor ratio. Its transpose the Rybczynski gives the components of the directional derivative of endowment changes onto output changes.

5. Empirical Analyses

We begin this section with some simple comments about what we can and cannot observe. The input-output data consist of values denominated in current dollars. Hence we cannot observe prices and quantities independently. We follow the convention established by Leontief (1951, p. 72) himself, who noted, "In order to obtain the corresponding physical amounts of all commodities and services, we simply define the unit of physical measurement of every particular type of product so as to make it equal to that amount of the commodity which can be purchased for one dollar at prevailing prices." The direct factor uses in each sector are measured in person-years for different categories of labor and in current dollars for the stocks of capital. Hence we observe physical quantities of labor, but we do not observe factor prices. We measure capital as the stock of fixed assets in each sector, measured in current dollars; hence this measure is fundamentally different from that for labor since it depends upon current prices. Again, we observe stocks of capital but not rates of return.

The input-output data are published by the Department of Commerce's Bureau of Economic Analysis (BEA). We use data that are disaggregated into 63 sectors.[5] The sum of each column of the input-output matrix is the gross industry output in each industry measured in millions of current dollars. The data on direct factor uses for capital are from the BEA and those for labor are from the Department of Labor's Bureau of

[5] The data are available at this URL http://www.bea.gov/industry/io_annual.htm

Labor Statistics (BLS).[6] We normalize these data by dividing every element in a column by gross industry output. Hence we measure the direct and indirect factor requirements needed for one million dollars of industry output. In essence, we have set the price of each unit of output to $1 million or the corresponding physical quantity to the amount that can be purchased for $1 million. The distinction between price and quantity is important since we consider separately the impact of endowment changes on the quantity of outputs, and the impact of price changes on factor payments.

Let B be the $f \times \ell$ matrix of direct factor inputs per unit of gross output, and C be the corresponding the $\ell \times \ell$ be matrix of intermediate inputs. $A = [B(I - C)^{-1}]^T$ is the $\ell \times f$ matrix of direct and indirect factor inputs used in our empirical analysis. The zero-profit condition implies that factor payments w satisfy:

$$Aw = \underset{\ell \times 1}{1}$$

There is an interesting interpretation of this condition in when $\ell > f$. In this case, the unit vector will almost surely not lie in the column space of A. In fact, this is exactly the situation of an econometrician trying to find the best fit for a left-hand variable (our unit vector of assumed output prices) onto the column space of the explanatory variables (the direct and indirect factor uses in every sector). We will use this intuition in the next sub-section to describe how the model with six mobile factors fits the data.

Since factor prices satisfy $w = A^+ \underset{\ell \times 1}{1} + (I - A^+ A)z$, we may write

$$\hat{p} = AA^+ \underset{\ell \times 1}{1}$$

[6] The data on labor inputs are available at this URL http://www.bls.gov/oes/current/oessrci.htm The Appendix describes exactly how we construct these aggregates.

This vector is the best estimate of the prices that are consistent with the zero-profit conditions that underpin our analysis. The projection matrix AA^+ maps any $\ell \times 1$ vector into the closest vector in the column space of the direct and indirect factor requirements. Hence, if $\ell > f$, then our best estimate of the national revenue function is

$$r(\hat{p}, v) = \hat{p}^T {A^+}^T v \tag{4}$$

and the shadow of the factors is $r_v(\hat{p}, v)^T = A^+\hat{p} = A^+ \underset{\ell\times 1}{1}$ since $A^+AA^+ = A^+$.

A. All Factors are Mobile

We first analyze a model where all factors are mobile and the number of sectors $\ell = 63$ exceeds the number of factors $f = 6$. Since changes in endowments are unrestricted in this case, we prefer to interpret the elements of ${A^+}^T$ as the Rybczynski derivatives. Each column of this matrix reports the impact of an increase in an endowment on the economy's vector of outputs under the assumption that goods prices and thus factor rewards are constant. Each column sum (of this transposed matrix) gives the shadow value of the factor in question.

For example, consider adding one additional Management and Technical person year to the economy's fixed resources. The column sum of the Rybczynski matrix shows that the shadow value of this worker is $\$0.145 \times 10^6$, and the logic inherent in (4) indicates that this sum is an estimated annual salary. The services of our hypothetical new worker will be distributed throughout the economy, and other factors will be reallocated to maintain constant factor proportions within sectors. The reported change in output in each sector then reflects this complete re-allocation of resources. In Heckscher-Ohlin theory, these effects are an indication of revealed comparative advantage. If domestic absorption is a fixed share of world production, those sectors whose outputs increase most will also contribute most to net exports. Rybczynski effects thus capture the impact of changes in endowments on the pattern of trade.

There is no easy way to report a table of $378 = \ell \times f = 63 \times 6$ numbers. In fact, no one has ever calculated an actual Rybczynski matrix before. Table 1 follows the tradition in trade theory and reports the strongest positive effect for each factor.

Table 1: Strongest Positive Rybczynski Effects

An increase in one unit of this factor:	**Increases output most in:**	**Change**
Capital	Real estate	39
Management and Technical Occupations	Computer systems design and related services	75
Education and Health Care Occupations	Educational services	21
Food Service and Maintenance Occupations	Food services and drinking places	31
Sales and Clerical Occupations	Retail trade	37
Production and Transportation Occupations	Transit and ground passenger transportation	20

Note: Capital is measured in millions of dollars. All other factors are measured in person years. Output effects are in thousands of dollars per year.

A million dollar increase in capital will increase output in real estate by $39 thousand, its strongest effect in any sector. Indeed, the capital intensity of real estate is the highest across all sectors in the economy; it employs $9 of capital per dollar of output.

Table 2: Strongest Negative Rybczynski Effects

An increase in one unit of this factor:	Decreases output most in:	Change
Capital	Computer systems design and related services	-13
Management and Technical Occupations	Retail trade	-30
Education and Health Care Occupations	Computer systems design and related services	-7
Food Service and Maintenance Occupations	Retail trade	-6
Sales and Clerical Occupations	Food services and drinking places	-10
Production and Transportation Occupations	Retail trade	-12

Note: Capital is measured in millions of dollars. All other factors are measured in person years. Output effects are in thousands of dollars per year.

Table 2 reports the strongest negative effect for each factor. For example, an increase in one Management and Technical person-year will decrease output most in Retail Trade. These detailed effects show a much richer and more varied picture than is typical in the literature. Hence our work stands in stark contrast to the usual approach that reports econometric estimates of output effects; good examples of this kind of work are Leamer (1984) and Harrigan (1995).

Such studies face serious data limitations; hence they focus on a more narrow range of sectors, usually only manufacturing outputs. Providing an apt comparison, we present further detail on the Rybczynski effects within the nineteen manufacturing industries in our data. Both Harrigan and Leamer conclude that capital has a positive effect on all manufacturing sectors, and this it is a source of comparative advantage in all manufacturing sectors. Table 3 presents our findings, and it identifies eight out of nineteen sectors whose output actually decreases. For example, an extra millions dollars of capital decreases the output of furniture and related products by $4 thousand.

Table 3: Capital's Rybczynski Effects on the Manufacturing Sectors

Sector	**Increase**
Food and beverage and tobacco products	3
Textile mills and textile product mills	0
Apparel and leather and allied product	-4
Wood products	-2
Paper products	2
Printing and related support activities	-2
Petroleum and coal products	14
Chemical products	2
Plastics and rubber products	0
Nonmetallic mineral products	1
Primary metals	3
Fabricated metal products	-1
Machinery	0
Computer and electronic products	0
Electrical equipment, appliances, and components	-1
Motor vehicles, bodies and trailers, and parts	0
Other transportation equipment	-1
Furniture and related products	-4
Miscellaneous manufacturing	-2

Note: Capital is measured in millions of dollars. Output effects are in thousands of dollars per year.

It is also interesting to examine the impact of an increase of one person-year of highly skilled labor (Professional Occupations) and unskilled labor (Production and Transportation Occupations) on manufacturing output. These effects are described in Table 4. In contrast to the limited impact of skilled labor on only two of ten industries reported by Harrigan (1995), we find a positive impact on fourteen industries. Indeed, most of the industries in which capital had a negative or neutral impact, such as computer and electronic products, and other transportation equipment, show a very strong positive impact from an increase in this kind of labor. This empirical finding is reassuring, since it suggests that the United States has a revealed comparative advantage in these sectors if indeed it is relatively abundantly endowed with highly skilled labor. Notice that many manufacturing industries—such as apparel and furniture--actually are more strongly affected by unskilled labor than skilled labor. Again, these rich Rybczynski effects show the importance of human capital even in traditional manufacturing sectors.

Table 4: Labor's Rybczynski Effects on the Manufacturing Sectors

Manufacturing Sectors	**Skilled Labor**	**Unskilled Labor**
Food and beverage and tobacco products	0	3
Textile mills and textile product mills	1	10
Apparel and leather and allied product	-1	16
Wood products	-2	9
Paper products	1	5
Printing and related support activities	1	8
Petroleum and coal products	1	-2
Chemical products	8	1
Plastics and rubber products	5	7
Nonmetallic mineral products	0	7
Primary metals	0	6
Fabricated metal products	5	9
Machinery	11	6
Computer and electronic products	28	1
Electrical equipment, appliances, and components	9	6
Motor vehicles, bodies and trailers, and parts	6	5
Other transportation equipment	22	3
Furniture and related products	1	12
Miscellaneous manufacturing	6	6

Note: Factors are measured in person years. Output effects are in thousands of dollars per year.

B. The Model's "Statistical" Fit

We now draw our attention to the model's overall fit. As we have emphasized, there are important theoretical and empirical implications from assuming that that $\ell > f$. Since the Stolper-Samuelson matrix is the transpose of the Rybczynski matrix, the values in Tables 1 and 2 also represent the sector which is the best friend and worst enemy of a given factor.[7] They actually report the impacts on the factor payment of a change in the price of that sector's output. However, if we consider an arbitrary price change in a single sector, we need to map it into the column space of the technology matrix using the idempotent (projection) matrix AA^{+}. Again, price changes are restricted to be directional derivatives that lie in the column space of A. This is exactly the situation that an econometrician faces who is trying to fit an arbitrary vector of data onto the column space of some explanatory variables. Our "data" are the assumed price vector $p = \underset{\ell\times 1}{1}$ and our explanatory variables are the factor uses in every sector, without a constant term. Our estimated coefficients are the shadow values of the factors we are analyzing.

Table 5 presents the results of this simple "estimation." Capital is measured in millions of dollars, so the "estimated reward" of $136,408 represents an economy-wide gross rate of return of 13.6%. All other factors are measured as person years, so the estimated coefficients are annual salaries. We find all six shadow values are significantly

[7] The best friend of a factor is the good whose marginal price effect on that factor's reward is maximal. In our theory, it is the index corresponding to maximal element of a row of A^{+}. The worst enemy of a factor is analogously the good whose marginal price effect on a factor's reward is minimal (and usually negative). A classic reference is Ethier (1984).

different from zero for a test of size 5%.[8] However, Education and Health Care Occupations has an estimated wage that is negative.

This negative shadow value is the best indication that something is amiss in reconciling the *direct* factor requirements for the different sectors with the input-output data on intermediate inputs. Table 5 gives the factor prices that best fit output process under the assumption that payments to capital and the five types of labor exhaust value added. If we had not tried to use an exhaustive list of factors, then the zero-profit conditions that are at the heart of the estimation would not be germane. We could not predict any factor price because the unobserved vector of costs imposes no discipline on a model that is based on the definition of unit-value isoquants. Much of the literature on trade theory that measures factor content makes this mistake.

Table 5: OLS Estimates of Factor Rewards

Factor	**Reward**	**Newey-West Standard Errors**
Capital	$136,408**	$25,939
Professional Occupations	$145,019*	$56,833
Education and Health Care Occupations	-$28,744**	$6,315
Food Service and Maintenance Occupations	$21,458*	$8,908
Sales and Clerical Occupations	$64,733**	$21,540
Production and Transportation Occupations	$46,829**	$9,696

Note: Capital is measured in millions of dollars. All other factors are measured in person years. The regression $R^2 = 0.935$ *and the number of observations* $n = 63$.
** denotes significance for a test of size 5%*
*** denotes significance for a test of size 1%*

[8] All the standard errors reported in this paper have been adjusted for heteroscedasticity using the Newey-West correction with $k = 3$.

A theoretical purist might assert stubbornly that there is nothing wrong with a model that predicts a negative shadow value for some factor. The value added in each sector includes information about indirect business taxes. It is easy to show in a simple model where all producer prices are positive that some factor rewards may be negative because the Stolper-Samuelson affects magnify tax wedges. The importance of our analysis is to show that the negative shadow value for education and healthcare occupations is no statistical fluke.

In fact, our technique of deriving the national revenue function and then estimating the shadow value of a factor is a way of confirming the validity of an important aspect of national income accounting. It is a commonplace that one cannot use the product approach to measure the services produced by many public sector employees. We tell our students in introductory courses in macroeconomics that the "output" of a policeman or a public school teacher corresponds exactly to what that worker earns. This accounting fiction maintains the identity between the income and the product approaches in national accounts. But it is quite a different exercise to ask the question, "What is the value of another public service employee, taking all factor prices including the pattern of indirect taxation as given?" The ruthless logic of the Rybczynski theorem reminds us that every extra employee must draw off resources from other sectors. Given how direct factor requirements have been measured, there is no guarantee that the overall effect of the reallocation of resources in the economy will be positive too.

Table A1 in the appendix shows the five types of labor that are aggregated into the rubric "Education and Health Care Occupations." To understand the seeming anomaly of a negative shadow value, we examined a more detailed model in which all twenty-two types of labor and also capital were mobile between sectors. In that case, the shadow values of only two types of labor in the rubric "Education and Health Care Occupations" are quite negative: a person-year of "Community and Social Services Occupations" is worth $-\$0.894 \times 10^6$; and a person-year of "Healthcare Practitioners and Technical Occupations" has value $-\$1.546 \times 10^6$. In the final subsection, we will show some evidence that indicates that the negative shadow value of Community and Social Services Occupations may have to do with a highly negative return on sector-specific capital used in "Social Assistance." In essence, a worker in social services actually uses a lot of sector-specific subsidies.

Again using the detailed taxonomy, we see that that the shadow value of a "Healthcare Support Occupations" (grouped with "Food Service and Maintenance Occupations") is $\$3.081 \times 10^6$, much higher than the salary of all the other occupations in that broad rubric. The correlation between the direct and indirect factor uses of "Healthcare Practitioners and Technical Occupations" and "Healthcare Support Occupation" across the 63 sectors in the American economy is an extraordinarily high 0.97. It is the highest among all $231 = 22 \times 21/2$ factor pairs. (By way of comparison, the analogous correlation between Capital and Management is -0.12, and the average of all such correlations in the American economy is 0.08.) This is a strong indication that these two types of labor are complementary within all the sectors in the economy, and it implies that either factor exhibits a very strong magnification effect. Perhaps it is reasonable to state that an increase in a *matched pair* of the detailed occupations "Healthcare Practitioners and Technical Occupations" and "Healthcare Support Occupations" has a shadow value of $\$3.081 \times 10^6 - \$1.546 \times 10^6 = \$1.535 \times 10^6$.

Consider the 63 "predicted" output prices $\hat{p} = AA^{+}1$. Imposing the assumption that they are independently distributed, we performed a likelihood ratio test based on the null hypothesis that all the prices were unity.[9] When there are only five mobile factors, we reject this hypothesis for a test of size 5%; the marginal significance level is 0.037. Following Leontief (1951), almost every scholar working with input-output accounts has imposed the (often implicit) normalization that the price of each sector's output is unity. To the best of our knowledge, no one has ever actually tested whether a model with several mobile factors is logically consistent. This test is perhaps one of our more important empirical contributions.[10]

Consider a given vector of endowments v measured without error. Since output prices must lie in the column space of the technology matrix, the model gives a "best estimate" for national revenue as described in (4). Our calculations show:

$$r(\hat{p}, v) = \$10.86 \times 10^{12}.$$

Actual GDP in 2003 was $\$11.11 \times 10^{12}$. Thus aggregation bias when we decided to use five broad types of labor for expositional simplicity has caused us to underestimate GDP by 2.3%.[11] The reader who is uncomfortable with pseudoinverses might prefer to think of our having estimated the entire national revenue function up to six parameters: the six shadow values for factors reported in Table 5 are coefficients from a regression of goods prices on factor uses without a constant term. Although our estimate "misses" actual national revenue slightly, it still describes 378 Rybczynski effects that are quite plausible.

[9] Our test statistic was $(1/\hat{\sigma}^2)\,[\sum_i (p_i - 1)^2 - \sum_i (\bar{p} - 1)^2]$, where $\hat{\sigma}^2$ is the maximum likelihood estimate of the population variance of the prices and $\bar{p}$ is the sample mean.

[10] In the case with 22 types of labor, the test statistic had a p-value of 0.25, and aggregation at that level does not seem to cause undue statistical mischief.

[11] When we redo the calculations with 22 types of labor, we underestimate GDP in 2003 by only 0.8%.

The categories for value added that actually appear in the input-output table itself are compensation for employees, gross operating surplus, and taxes. Fisher and Marshall (2008a) use these data to define factor usages for each of forty-eight sectors for thirty-three different OECD countries. Every such technology matrix is row stochastic because it reports simply cost shares for each industry. That model fits perfectly because then $Aw = p$ has an exact solution for $p = 1$ in which $w = 1$ too. In that case, there is no difference between estimated and actual GDP. We found this rarefied version of the model less than satisfactory for our work here for three reasons. First, we wanted to follow the mainstream of the literature by reporting effects having to do with several different kinds of labor measured in physical units of person-years. This is the "natural" approach, and it is much more illuminating than focusing on only one type of aggregate labor. Second, it is not very interesting to report that the shadow value for a dollar's worth of any input is automatically one dollar; hence our current approach allows us to "estimate' the shadow values of factors in an interesting way. Third, we wanted to remind the profession that the usual practice of incorporating data on factor usages from different sources than the input-output table itself imposes some cost in terms of the model's own logical consistency. We are happy to report that the model with disaggregated labor performs well enough from the perspective of national income accounting in the closed economy.

We can also use the estimated national revenue function to predict the economy's output vector. This is analogous to the typical in-sample predictions that an econometrician might perform. We are interested in

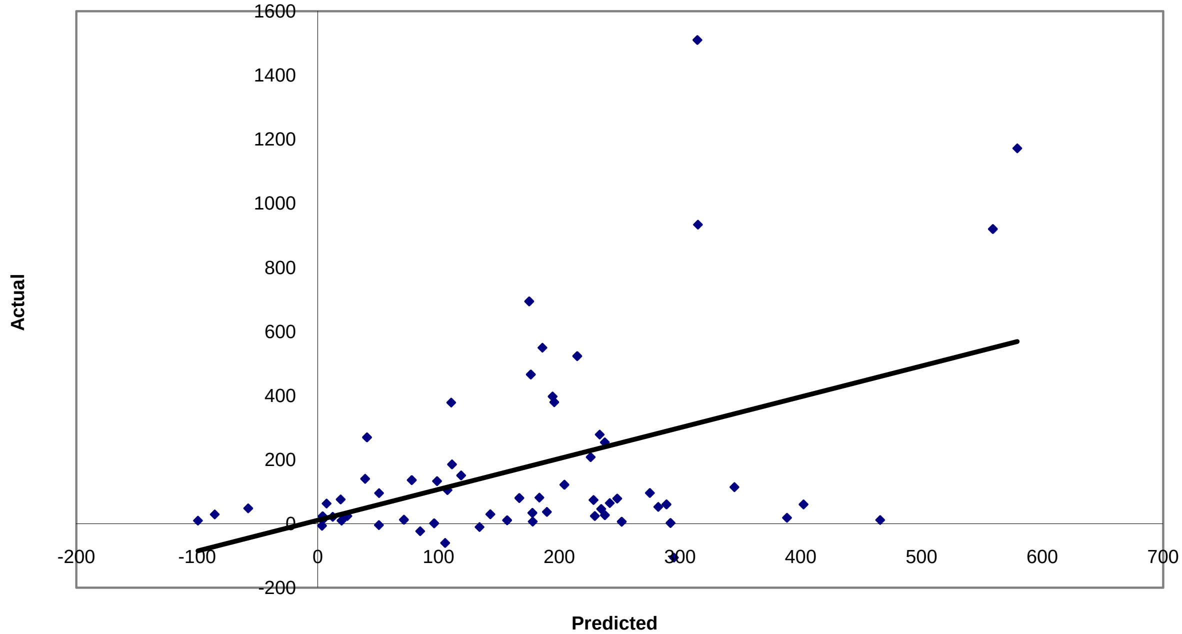
$$\hat{y} = r_p(\hat{p}, v)^{\,T} = A^{+^T} v$$

There is nothing in the construction of the Rybczynski matrix that guarantees that the actual outputs by sector will be the vector of minimum norm that satisfies the full employment conditions. The logic inherent in Fig. 2 shows that any actual output vector can be decomposed into two parts: one that matters for national revenue and another that moves along the economy's production possibility frontier. One can think of these two components as a variance decomposition of disaggregated outputs for the economy. It is natural to ask, "What fraction of the variance in actual output is predicted by the model?"

The mathematical formulation of this question is, "What is the ratio $\|\hat{y}\|/\|y\|$?" Since the unexplained part of output u in Fig. 2 is orthogonal to $\hat{y}$, this is exactly analogous to a traditional measure of goodness of fit in a regression where the data have mean zero. The variance decomposition for the actual vectors of disaggregated output in 2003 shows that our model explains a fraction 0.47 of the variability in the data. This goodness of fit is quite solid for a "regression" using cross-sectional data.

Fig. 3 gives the scatter plot of our predictions against the in-sample values. It is obvious the model fits well enough and that there may be heteroskedasticity in our data.

Fig. 3: Output by Sector in 2003
Billions of Current Dollars



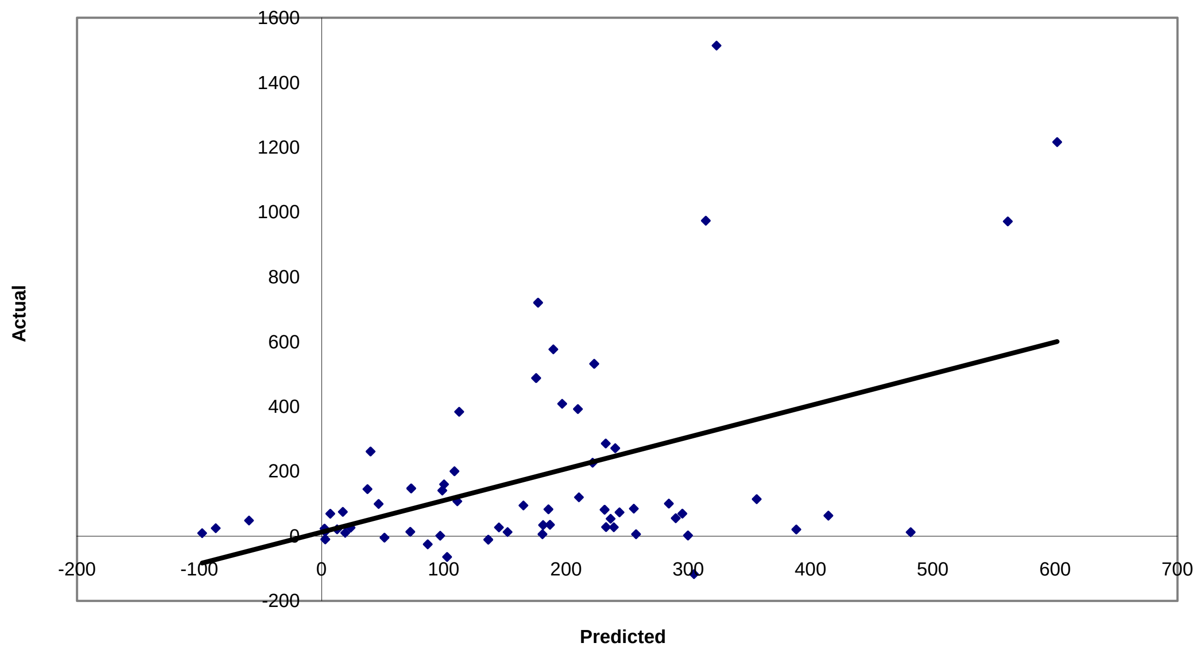
The (usual and prosaic) regression line shown in this figure is represented by

$$y_i = \underset{(40726)}{10431} + \underset{(0.314)}{0.963}x_i,\ R^2 = 0.2, n = 63$$

where the asymptotic standard errors in parentheses are adjusted for heteroscedasticity using the Newey-West corrections. The natural F-test imposes the two restrictions that the constant is zero and the slope is unity. Its statistic has a value of 0.035, with an asymptotic p-value of 0.982. Hence, the model predicts well in sample.

We also present an out of sample prediction for 2004, using the pseudoinverse of the 2003 technology matrix and the endowment vector for 2004. We hold prices constant by deflating each sector's output in 2004 by an industry level price index. Figure 4 depicts the results of this exercise.

Fig. 4: Output by Sector in 2004
Billions of 2003 Dollars



The line in this figure is represented by

$$y_i = \underset{(41548)}{12041} + \underset{(0.318)}{0.978}x_i,\ R^2 = 0.2, n = 63$$

The natural F-statistic has a value of 0.056, with an asymptotic p-value of 0.978. Hence, the model predicts very well out of sample too.

The main point of this sub-section is that we have a very good mapping from the endowment vector to outputs by sector. It is not a magic bullet because the model fit is far from perfect. But the logic inherent in the Rybczynski effects depicted in Figure 2 are borne out by our analysis of data from the American economy, both in sample in 2003 and out of sample in 2004. This is the first time anyone has ever computed an entire Rybczynski matrix. In essence, we are bringing life to an older literature in trade theory that died from a lack of empirical relevance.

C. Sector-Specific Capital

This subsection allows us to show the empirical power of our general theoretical approach. The specific factors model—also called the Ricardo-Viner Model--has an important place in trade theory and in applied general equilibrium studies. It is particularly apt for doing comparative statics because the national revenue function is well behaved. It is also used in the study of the political economy of taxation since the effects of distorting taxes on specific factors are simple to model.

Now the transpose of the technology matrix has this form:

$$A^T = \begin{bmatrix} b_{K(1),1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & b_{K(\ell),\ell} \\ b_{11} & \dots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{51} & \dots & b_{5\ell} \end{bmatrix} [I - C]^{-1}$$

Where $b_{K(i),i}$ is the direct unit input requirement of specific factor $K(i)$ in the $i-th$ sector for $i \in \{1, \dots, 63\}$ and b_{ij} is the direct unit input requirement of mobile factor i in sector j. C is again the economy's input-output matrix. The five mobile factors are the different kinds of labor that we are analyzing, and the 63 specific factors are the measured uses of capital in each sector. Thus we are assuming that capital is not mobile, and there is no such thing as an economy-wide rate of return on capital. Notice that this model has more factors than goods since $f = 68 > 63 = \ell$. Now our preferred interpretation of A^+ is as a Stolper-Samuelson matrix. It has $f \times \ell = 68 \times 63 = 4284$ elements. Each measures the effect of increasing the price of some good on a factor's reward. Of course, there is no simple way to summarize all these numbers

The properties of the pure Ricardo-Viner model are well known.[12] For example, an increase in the price of the $i-th$ good will raise the return of the specific factor in that sector. But there is an important subtlety in empirical work. The technology matrix actually incorporates the direct and indirect uses of all factors. So every sector requires the use of every factor—mobile and specific—because of the effect that intermediate inputs have on factor content. Hence, it is not the case that an increase in the price of a sector will automatically lower the return to the specific factors used in all the other sectors. In fact, it is not even true that an increase in the price of one sector will have its strongest impact on that sector's specific factor. In our data, an increase in the price of real estate actually has its strongest effect on the reward for specific factor called "capital used in educational services."[13] In every other case, the strongest effect of a price increase is on the specific factor used in that sector.

We would like to reiterate an important theoretical observation that arose in the discussion of the Solow model as the leading example of the case where there are more factors than goods. The Stolper-Samuelson effects we report *are not* the derivatives of the national revenue function of a model with mobile factors and a smooth neo-classical production function in each sector; hence, they do not correspond to the textbook treatment of the comparative statics of this model. We have been very explicit about holding factor uses constant when output prices change; that is why we derived the national revenue function for a Leontief technology and then appealed to the envelope theorem to assert that our results were germane the effects of price changes in a more general setting. In the usual treatment of the Ricardo-Viner model, the mobile factor

[12] Feenstra (2003) has a very nice exposition.
[13] A plausible conjecture is that local school districts capture the effects that pricier real estate has on the tax base for local expenditures.

flows into the sector whose price has increased. If output in that sector changes at all, then the unit input requirements of the fixed factor must change too. Hence the textbook model really captures two effects: (1) that of a price change; and (2) that of an ancillary reallocation of resources between sectors. Only the first one is a true Stolper Samuelson effect, and that is what we report here. Also, only the first effect has a natural interpretation as the dual of a Rybczynski derivative. We feel that our analysis is in keeping with the spirit of traditional trade theory.

Tables 6 and 7 present the Stolper-Samuelson effects on the five mobile factors. The table lists five labor aggregates, with the first row corresponding to most skilled category and the last row to the least skilled. Now a factor's best friend is the sector whose price increase has the greatest positive effect on its reward, and its worst enemy is the sector with the greatest negative such impact. Table 6 shows that skilled labor's best friend is the sector called "Computer Systems Design and Related Services," and Table 7 shows that skilled labor's worst enemy is "Legal Services." The best friend of the least skilled labor (Production and Transportation Occupations) is "Furniture and Related Products," and its worst enemy is Construction.

Table 6: Positive Stolper-Samuelson Effects on the Mobile Factors

The best friend of this factor:	Is this sector	Reward
Professional Occupations	Computer systems design and related services	114
Education and Health Care Occupations	Legal services	95
Food Service and Maintenance Occupations	Food services and drinking places	41
Sales, Clerical, and Construction Occupations	**Legal services**	**89**
Production and Transportation Occupations	Furniture and related products	50

Note: Output prices are measured in millions of dollars. Factor rewards are measured in thousands of dollars per year.

Table 7: Negative Stolper-Samuelson Effects on the Mobile Factors

The worst enemy of this factor:	Is this sector	Reward
Professional Occupations	Legal services	-88
Education and Health Care Occupations	Construction	-31
Food Service and Maintenance Occupations	Legal services	-43
Sales, Clerical, and Construction Occupations	Social assistance	-20
Production and Transportation Occupations	Construction	-31

Note: Output prices are measured in millions of dollars. Factor rewards are measured in thousands of dollars per year.

The interpretation of the magnitudes of the numbers reported in these tables is straightforward. Imagine a million dollar increase in the price of some sector's output. Then the Stolper-Samuelson effect is an increase to the reward for a person-year of the factor in question. Thus a million dollar increase in net output of Computer Systems Design and Related Services increases the reward to professional occupations by $114 thousand per year. Likewise, a million dollar increase in the price of legal services decreases the reward to professional occupations by $88 thousand per year.

We conclude the empirical analysis in this subsection with an examination of the effects that changes in output prices have on some specific factors. We examine the four largest sectors in our data and also the sector called Petroleum and Coal Products because the effect of an oil price shock is topical. Table 8 reports our results. It is typical that a price increase raises the rate of reward of the specific factor used in that sector, and some sectors exhibit a magnification effect for their own specific factors. It is also interesting to note that there are moderately strong negative effects of output price increases to the rewards of some specific factors used in other sectors of the economy. For example, a million dollar increase in the price of retail trade lowers the reward to sector-specific capital in Construction by $489 thousand.

Table 8: Selected Stolper-Samuelson Effects on Some Specific Factors

This Sector (GDP Share)	Maximal Effect on Capital Specific to this Sector	Reward	Minimal Effect on Capital Specific to this Sector	Reward
Real estate (11%)	Educational services	205	Legal services	-293
State and local general government (10%)	State and local general government	304	Miscellaneous professional, scientific, and technical services	-4
Retail trade (8%)	Retail trade	1340	Construction	-489
Food and beverage and tobacco products (3%)	Food and beverage and tobacco products	2743	Furniture and related products	-208
Petroleum and coal products (1%)	Petroleum and coal products	2299	Truck transportation	-128

Note: Output prices are measured in millions of dollars. Factor rewards are measured in thousands of dollars per year.

In this case, it is not possible to describe how the model of the specific factors actually fits the data because *it fits them perfectly*. In the Ricardo-Viner model, we have sufficiently many free parameters--the shadow values of all the economy's 68 factors--to fit the hypothesized prices perfectly. The predicted outputs by sector are exactly equal to the actual outputs, output prices are identically $p = 1$, and predicted national revenue is exactly equal to actual national revenue in 2003. Hence, Leontief's system of input-output accounting is completely consistent if one is willing to define at least one factor that is specific to each sector. In essence, this assumption allows one to calculate sector-specific rates of return for each type of measured capital, and one can fit the factor cost equations identically. The consequence of all this mathematical purity is that factor prices are not uniquely determined. The Stolper-Samuelson matrix picks out the unique vector of factor prices that are orthogonal to the unit cost functions for all active sectors. Then Shephard's Lemma guarantees that factor demands will be consistent with actual endowments and thus full employment. Still, the technology matrix is not independent of endowments in general. We think this is a small price for using the power of general equilibrium in characterizing the economy's supply side.

D. The Shadow Values of the Mobile Factors in the Model of Specific Factors

We have used the Ricardo-Viner model to describe the Stolper-Samuelson effects in the national economy. These effects are exact; they are not estimates. The particular solution $\hat{w} = A^{+}p = A^{+}1$ is the vector of minimum norm in the space of factor prices that is consistent with the zero-profit conditions. All possible factor prices are:

$$w = A^{+}1 + (I - A^{+}A)z$$

where z is arbitrary. Since $v^{T}(I - A^{+}A)z = 0$, we can interpret $\Delta w = (I - A^{+}A)z$ as the set of all possible changes in factor prices such that $v^{T}\Delta w$. Since each element of the endowment vector is positive, it is not possible for every element of Δw to be strictly positive. Hence these factor price changes cannot be Pareto ranked; any movement away from the particular solution $\hat{w}$ benefits some factor only at the expense of another.

Still, the particular solution for factor prices corresponds with aggregate factor demands that maintain full employment. Hence the shadow values of the five mobile factors in the Ricardo-Viner model have economic significance, and they beg comparison with the model where all factors are mobile. Table 9 makes this contrast. (The corresponding rates of return to the 63 kinds of sector-specific capital are presented in Table A2 in the Appendix.) Since the shadow values in the Ricardo-Viner model are one of many possible solutions, we would not expect them to correspond with the estimates in our first model. In that case, some of the payments attributed to labor almost surely represented returns to sector-specific capital.

Indeed, Table 9 allows us to reconsider the most anomalous finding in that earlier exercise. For example, the shadow value of “Education and Health Care Occupations” in the Ricardo-Viner model is \$54,595; this is quite different from the statistically significant negative value -\$28,744 reported in Table 5. Table A2 in the Appendix shows large negative returns to capital specific to the two sectors Educational Services and Social Assistance, which together employ 46% of the 18.3 million workers in the broad rubric “Education and Health Care Occupations.” The most negative rate of return on capital specific to any sector is that in “Educational Services”, which costs the national economy \$2.08 million annually for every \$1 million increase in its stock. Likewise the rate of return on capital specific to Social Assistance is -46%. Perhaps the negative wage estimate in the first model confounded a negative return on sector-specific capital with the shadow value of a mobile factor. This finding may be reassuring to proponents of the view that teachers—not bricks and mortar—matter for educational reform that will have a positive effect on national income.

Table 9: The Shadow Values of Types of Labor in the Two Different Models

Factor	Ricardo-Viner	Mobile Factors
Management and Technical Occupations	$36,987	$145,019
Education and Health Care Occupations	$54,595	-$28,744
Food Service and Maintenance Occupations	-$2,799	$21,458
Sales and Clerical Occupations	$120,499	$64,733
Production and Transportation Occupations	$43,539	$46,829

6. Conclusion

This paper has made two main contributions. The first was theoretical, and the second was empirical. Our theoretical contribution was to show that the input-output accounts contain all the information necessary to describe the relationship between factor endowments and output supplies. Since the Rybczynski effects have to do with quantities and assume fixed output prices and factor rewards, this result is not so much of a surprise when it is stated at this level of generality. But it is quite startling that input-output accounts also contain complete information about the relationship between output prices and factor rewards. The duality between the Rybczynski and Stolper-Samuelson matrices is well understood. But no one has ever shown an explicit form for the national revenue function before. This applied theoretical contribution has important theoretical and empirical implications.

Our empirical contributions were to adumbrate some of the details of the supply side of the American economy in 2003. It is not much of a surprise that capital's gross rate of return was 13.6% in that year or that an million dollar increase in the costs of refined petroleum products has its strongest negative effect (-12.8%) on the reward to capital specific to Truck transportation. But the very plausibility of these results shows that our theory has the ring of truth behind it. No one has ever used the Moore-Penrose generalized inverse in applied general equilibrium theory before. Most pieces of mathematical software have an easy function that readily computes the unique Moore Penrose pseudoinverse of any non-null matrix. Now economists can use that function in many different applications.

Our work has advanced input-output accounting significantly by showing the exact relationship between these accounts and the national revenue function. Thus any scholar in macroeconomics interested in the wealth effects of supply-side shocks will find ready use for the techniques that we have developed here. For example, a neutral technology shock in a small open economy can be modeled as a parametric change in endowments. Then the national revenue function will show the exact output effects for a fixed vector of prices. Also, trade economists and development economists, who regularly work with input-output accounts, can now evaluate whether the levels of aggregation for factor inputs used in their work are innocuous. Indeed, economists might well treat disaggregated models with only two factors of production with some caution until they examine the exact properties of the corresponding national revenue function with greater care.

In sum, our work and the empirical tools we have created will have broad appeal to trade theorists, to macroeconomists, to development economists, to labor economists, and to public finance economists. Any field in our discipline that needs to explore the relationship between factor prices and factor rewards or that between resources and output supplies can build on our methods.

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Appendix

This Appendix serves two purposes. First, it supports some of the empirical observations in the text. Second, it gives the complete lists of factors and also of sectors that characterize our data. We urge the reader to glance at the rubrics in Tables A1 and A2 to get a sense of the level of aggregation that we have used.

The Bureau of Economic Analysis publishes annual input-output tables for the U.S. economy disaggregated into 65 sectors by the North American Industrial Classification System. Data on factor uses are published in a consistent manner with a few exceptions. The input output data describe four government sectors, including government enterprises for federal and for state and local government. To match endowment data, these four sectors were merged into two by combining the general government and government enterprises at the federal level and at the state and local level. Hence our study is based on 63 sectors, including two government sectors.

Data on labor factor inputs by sector are taken from the Bureau of Labor Statistics November 2003 Occupational Employment and Wage Estimates at URL http://www.bls.gov/oes/oes_2003_n.htm. These data are categorized by 2, 3 and 4 digit NAICS industry, but do not include the self-employed. The source of this data, the Occupational Employment Statistics survey, categorizes workers into about 770 categories which are aggregated into 22 broad classifications. In order to make these extensive data more manageable we have further aggregated the data into five categories that correspond to each of the two-digit rubrics summarized in Table A1. The labor data were easily matched to the input-output data since both follow the NAICS. However, the OES data did not include employment in the three digit NAICS classification 111 Farms, for which we substituted OES data for the three-digit NAICS 115 Support Activities for Agriculture and Forestry.

Data on capital inputs by sector are taken from the Bureau of Economic Analysis Table 3.1ES. Current Cost Net Stock of Private Fixed Assets by Industry for the private sector and from Table 7.1 B Current Cost Net Stock of Government Fixed Assets for the government sector. The out-of-sample prediction presented in Figure 4 computes the 2004 endowment vector based on the BLS labor input data described above in 2004. The capital stock for 2004 is computed by increasing each sector's capital in 2003 by the quantity index published in the Bureau of Economic Analysis, published in Table 3.2ES for the private sector and Table 7.2B for the government sector. Industry output for 2004 is taken from the BEA input output data for 2004, deflated by the industry level price indexes also published by the BEA.

Table A1: The Labor Aggregates

Classification	Description	Employment Share
Aggregate 1	**Professional Occupations**	**13.9%**
11-0000	Management occupations	5.1%
13-0000	Business and financial operations occupations	4.0%
15-0000	Computer and mathematical occupations	2.2%
17-0000	Architecture and engineering occupations	1.8%
19-0000	Life, physical, and social science occupations	0.9%
Aggregate 2	**Education and Health Care Occupations**	**14.4%**
21-0000	Community and social services occupations	1.3%
23-0000	Legal occupations	0.7%
25-0000	Education, training, and library occupations	6.2%
27-0000	Arts, design, entertainment, sports, and media occupations	1.2%
29-0000	Healthcare practitioners and technical occupations	4.9%
Aggregate 3	**Food Service and Maintenance Occupations**	**18.7%**
31-0000	Healthcare support occupations	2.5%
33-0000	Protective service occupations	2.3%
35-0000	Food preparation and serving related occupations	8.1%
37-0000	Building and grounds cleaning and maintenance occupations	3.4%
39-0000	Personal care and service occupations	2.3%
Aggregate 4	**Sales, Clerical, and Construction Occupations**	**37.6%**
41-0000	Sales and related occupations	10.6%
43-0000	Office and administrative support occupations	17.8%
45-0000	Farming, fishing, and forestry occupations	0.4%
47-0000	Construction and extraction occupations	4.8%
49-0000	Installation, maintenance, and repair occupations	4.1%
Aggregate 5	**Production and Transportation Occupations**	**15.3%**
51-0000	Production occupations	8.0%
53-0000	Transportation and material moving occupations	7.4%
	Total Employment (person years in 2003)	127,174,490

Table A2: Rates of Return on Sector-Specific Capital

Sector	Return
Farms	14%
Forestry, fishing, and related activities	70%
Oil and gas extraction	16%
Mining, except oil and gas	11%
Support activities for mining	11%
Utilities	13%
Construction	-103%
Food and beverage and tobacco products	41%
Textile mills and textile product mills	-7%
Apparel and leather and allied product	-15%
Wood products	-2%
Paper products	23%
Printing and related support activities	3%
Petroleum and coal products	34%
Chemical products	56%
Plastics and rubber products	26%
Nonmetallic mineral products	24%
Primary metals	19%
Fabricated metal products	18%
Machinery	18%
Computer and electronic products	21%
Electrical equipment, appliances, and components	51%
Motor vehicles, bodies and trailers, and parts	65%
Other transportation equipment	30%
Furniture and related products	3%
Miscellaneous manufacturing	49%
Wholesale trade	46%
Retail trade	-107%
Air transportation	12%
Railroad transportation	4%
Water transportation	19%
Truck transportation	32%
Transit and ground passenger transportation	-14%
Pipeline transportation	11%
Other transportation and support activites	-87%
Warehousing and storage	-27%
Publishing industries (includes software)	98%
Motion picture and sound recording industries	60%
Broadcasting and telecommunications	19%
Information and data processing services	67%
Federal Reserve banks and credit intermediation	32%

Securities, commodity contracts, and investments	92%
Insurance carriers and related activities	36%
Funds, trusts, and other financial vehicles	12%
Real estate	8%
Rental and leasing services and lessors of intangible assets	27%
Legal services	264%
Miscellaneous professional, scientific, and technical services	185%
Computer systems design and related services	-76%
Management of companies and enterprises	23%
Administrative and support services	-114%
Waste management and remediation services	8%
Educational services	-208%
Ambulatory health care services	45%
Hospitals and nursing and residential care facilities	-6%
Social assistance	-46%
Performing arts, spectator sports, museums, and related activities	35%
Amusements, gambling, and recreation industries	31%
Accommodation	3%
Food services and drinking places	95%
Other services, except government	27%
Federal government	25%
State and local government	11%

Note: Each return is the shadow value for national income of a stock of one million dollars of sector-specific capital.